STABILITY OF MOTION OF A FREE SOLID WITH A CAVITY COMPLETELY FILLED WITH A VISCOUS LIQUID IN THE FORCE FIELD OF TWO STATIONARY ATTRACTING CENTERS

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Sufficient conditions for the stability of the circular motion of the center of mass of a system consisting of a solid with a cavity completely filled with a viscous incompressible liquid are obtained; conditions for its relative equilibrium with respect to certain known parameters are derived. The investigation is carried out with the aid of Rumiantsev's formulation and by his method for solving problems on the stability of motions of liquid-containing solids [1].

In the case of one attracting center, sufficient conditions for the stability of a liquid-filled solid were obtained by Kolesnikov [2]. The same problem for the case of a solid alone was solved by Beletskii [3].

1. Let us consider the motion of a free mechanical system in the form of a solid with a cavity of arbitrary shape which is filled completely with a homogeneous viscous liquid and moves in the force field of two stationary attracting centers with inverse square attraction laws.

Let $O\xi_1\xi_2\xi_3$ be a stationary orthogonal coordinate system. The stationary attracting centers N_1 and N_2 of the masses m_1 and m_2 (one of these masses can be negative, in which case the corresponding center repels instead of attracting) and with the gravitational constants f_1 and f_2 are assumed to lie at the points with the coordinates $\xi_3 = \xi_3^{(1)}$ and $\xi_3 = \xi_3^{(2)}$ on the axis $O\xi_3$.

Let us introduce the two additional stationary orthogonal coordinate systems with their origins at the center of mass G of the system: 1) the system $Gx_1x_2x_3$ whose axis Gx_3 has the same direction as the axis $O\xi_3$, axis Gx_2 is perpendicular to the axis $O\xi_3$ and is directed from the point G towards the axis $O\xi_3$, and axis Gx_1 is perpendicular to the axes Gx_2 and Gx_3 and is directed in such a way that the systems of axes $O\xi_1\xi_2\xi_3$ and $Gx_1x_2x_3$ have the same orientation, and 2) the system $Gy_1y_2y_3$ whose axes are directed along the principal axes of the central ellipsoid of inertia of the system under consideration for the point G; the axes of this ellipsoid are also the principal axes of inertia of both the solid and the liquid.

Thus, if A_1, A_2, A_3, I_1, I_2, I_3, and J_1, J_2, J_3 are the principal moments of inertia relative to the axes y_1, y_2, y_3, respectively, of the solid-liquid system as a whole, then

$$A_1 = I_1 + J_1 \qquad (123)$$

The directions of the axes x_i ($i = 1, 2, 3$) and of the radius vectors R_1 and R_2 extending from the point G to the attracting centers N_1 and N_2 relative to the coordinate axes $Gy_1y_2y_3$

will be expressed in terms of the direction cosines α_{i1}, α_{i2}, α_{i3}, β_1, β_2, β_3, and γ_1, γ_2, γ_3, respectively.

Let r, ϕ, ξ_3 be the cylindrical coordinates of the center of mass G of the system; M_1 and M_2 are the mass of the solid and liquid, respectively; $M = M_1 + M_2$ is the total mass of the system.

The function of the forces acting on the system is given by the integral

$$U = \int_M \left(\frac{f_1 m_1}{\rho_{1m}} + \frac{f_2 m_2}{\rho_{2m}}\right) dm$$

Here

$$\rho_{1m}^2 = R_1^2 - 2R_1(y_1\beta_1 + y_2\beta_2 + y_3\beta_3) + y_1^2 + y_2^2 + y_3^2, \qquad R_1^2 = r^2 + (\xi_3^{(1)} - \xi_3)^2$$

$$\rho_{2m}^2 = R_2^2 - 2R_2(y_1\gamma_1 + y_2\gamma_2 + y_3\gamma_3) + y_1^2 + y_2^2 + y_3^2, \qquad R_2^2 = r^2 + (\xi_3^{(2)} - \xi_3)^2$$

We shall consider the problem in restricted formulation, i.e. we shall replace the above force function U by an approximation which we obtain by expanding the integrand in series in powers of y_1/R_i, y_2/R_i, y_3/R_i $(i = 1, 2)$, and by neglecting terms of higher than the second order of smallness. This yields

$$U = \sum_{i=1}^{2} \sigma_i \left[\frac{M}{R_i} - \frac{3}{2R_i^5}(A_1 y_1^{(i)2} + A_2 y_2^{(i)2} + A_3 y_3^{(i)2}) + \frac{A_1 + A_2 + A_3}{2R_i^3}\right]$$

$$(\sigma_i = f_i m_i, \ i = 1, 2)$$

Here $y_1^{(1)}$, $y_2^{(1)}$, $y_3^{(1)}$ and $y_1^{(2)}$, $y_2^{(2)}$, $y_3^{(2)}$ are the coordinates of the sttracting centers N_1 and N_2 in the system of axes $Gy_1y_2y_3$,

$$y_1^{(i)} = r\alpha_{21} + (\xi_3^{(i)} - \xi_3)\alpha_{31}, \qquad y_2^{(i)} = r\alpha_{22} + (\xi_3^{(i)} - \xi_3)\alpha_{32}$$

$$y_3^{(i)} = r\alpha_{23} + (\xi_3^{(i)} - \xi_3)\alpha_{33} \qquad (i = 1, 2)$$

We denote by Ω_1, Ω_2, Ω_3 and ω_1, ω_2, ω_3, respectively, the projections on the axes y_1, y_2, y_3 of the vectors of the absolute and relative (with respect to the coordinate system $Gx_1x_2x_3$) instantaneous angular velocity of the solid. Clearly,

$$\Omega_1 = \dot\varphi\alpha_{31} + \omega_1, \qquad \Omega_2 = \dot\varphi\alpha_{32} + \omega_2, \qquad \Omega_3 = \dot\varphi\alpha_{33} + \omega_3 \qquad (\dot\varphi = d\varphi/dt)$$

We now have the following expressions for the kinetic energy of the entire system T, that of the solid T_1, and that of the liquid T_2:

$$T = T_1 + T_2, \qquad 2T_1 = M_1(\dot r^2 + r^2\dot\varphi^2 + \dot\xi_3^2) + I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2$$

$$2T_2 = M_2(\dot r^2 + r^2\dot\varphi^2 + \dot\xi_3^2) + J_1\Omega_1^2 + J_2\Omega_2^2 + J_3\Omega_3^2 + 2(g_1\Omega_1 + g_2\Omega_2 + g_3\Omega_3) +$$

$$+ \rho\int_\tau (u_1^2 + u_2^2 + u_3^2)\, d\tau$$

Here

$$g_1 = \rho\int_\tau (y_2u_3 - y_3u_2)\, d\tau \qquad (123)$$

are the projections on the axes y_1, y_2, y_3 of the moment vector relative to the center of mass G of the system of momenta of the liquid particles in their motion relative to the coordinate axes $Gy_1y_2y_3$: u_1, u_2, u_3 are the projections on the same axes of the velocity vector of the liquid particles relative to the coordinate axes $Gy_1y_2y_3$; ρ is the density of the liquid; τ is the volume of the cavity.

We can now write the equations of motion of the mechanical system,

$$M(\ddot r - r\dot\varphi^2) = \frac{\partial U}{\partial r}, \qquad M\ddot\xi_3 = \frac{\partial U}{\partial \xi_3}$$

$$\frac{d}{dt}[Mr^2\dot\varphi + (A_1\Omega_1 + g_1)\alpha_{31} + (A_2\Omega_2 + g_2)\alpha_{32} + (A_3\Omega_3 + g_3)\alpha_{33}] = 0 \qquad (1.1)$$

$$A_1\frac{d\Omega_1}{dt} + (A_3 - A_2)\Omega_2\Omega_3 + \frac{dg_1}{dt} - \Omega_2 g_3 - \Omega_3 g_2 = L_1 \qquad (123) \qquad (1.2)$$

$$\frac{d}{dt}(V_1 + \Omega_2 y_3 - \Omega_3 y_2 + u_1) + \Omega_2 (V_3 + \Omega_1 y_2 - \Omega_2 y_1 + u_3) - \Omega_3 (V_2 + \Omega_3 y_1 - \Omega_1 y_3 + u_2) =$$

$$= F_1^{(1)} + F_1^{(2)} - \frac{1}{\rho}\frac{\partial p}{\partial y_1} + \nu \Delta u_1 \quad (123) \qquad (1.3)$$

$$\frac{d\alpha_{21}}{dt} = \Omega_3 \alpha_{22} - \Omega_2 \alpha_{23} + \varphi^{\cdot} (\alpha_{32}\alpha_{23} - \alpha_{33}\alpha_{22}) \quad (123), \qquad \frac{d\alpha_{31}}{dt} = \Omega_3 \alpha_{32} - \Omega_2 \alpha_{33} \quad (123)(1.4)$$

Here L_1, L_2, L_3 are the projections on the axes y_1, y_2, y_3 of the moments of the external forces, which in our approximation are of the form

$$L_1 = \frac{3\sigma_1}{R_1^3}(A_3 - A_2)\beta_2\beta_3 + \frac{3\sigma_2}{R_2^3}(A_3 - A_2)\gamma_2\gamma_3 \quad (123)$$

V_1, V_2, V_3 are the projections on the axes y_1, y_2, y_3 of the vector of the velocity of the center of mass G of the system; $F_1^{(1)}$, $F_2^{(1)}$, $F_3^{(1)}$ and $F_1^{(2)}$, $F_2^{(2)}$, $F_3^{(2)}$ are the projections on the same axes of the attractive forces exerted on a liquid particle by the attracting centers N_1 and N_2, respectively; p is the hydrodynamic pressure; $\nu = \mu/\rho$ is the kinematic coefficient of viscosity; μ is the coefficient of viscosity; Δ is the Laplacian.

To Eqs. (1.1) to (1.4) we must add the continuity Eq.

$$\frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} + \frac{\partial u_3}{\partial y_3} = 0 \qquad (1.5)$$

the boundary conditions $u_1 = u_2 = u_3 = 0$ at the walls S of the cavity, and the relations

$$\beta_1 = \frac{r}{R_1}\alpha_{21} + \frac{\xi_3^{(1)} - \xi_3}{R_1}\alpha_{31}, \quad \beta_2 = \frac{r}{R_1}\alpha_{22} + \frac{\xi_3^{(1)} - \xi_3}{R_1}\alpha_{32}, \quad \beta_3 = \frac{r}{R_1}\alpha_{23} + \frac{\xi_3^{(1)} - \xi_3}{R_1}\alpha_{33}$$

$$\gamma_1 = \frac{r}{R_2}\alpha_{21} + \frac{\xi_3^{(2)} - \xi_3}{R_2}\alpha_{31}, \quad \gamma_2 = \frac{r}{R_2}\alpha_{22} + \frac{\xi_3^{(2)} - \xi_3}{R_2}\alpha_{32}, \quad \gamma_3 = \frac{r}{R_2}\alpha_{23} + \frac{\xi_3^{(2)} - \xi_3}{R_2}\alpha_{33}$$

The equations of motion of mechanical system (1.1) to (1.5) enable us to write the energy relation

$$\frac{d}{dt}(T - U) = -\mu \int_\tau \left[2\sum_{i=1}^{3}\left(\frac{\partial u_i}{\partial y_i}\right)^2 + \left(\frac{\partial u_2}{\partial y_3} + \frac{\partial u_3}{\partial y_2}\right)^2 + \left(\frac{\partial u_3}{\partial y_1} + \frac{\partial u_1}{\partial y_3}\right)^2 + \left(\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right)^2\right] d\tau \qquad (1.6)$$

and the following first integrals:

$$Mr^2\varphi^{\cdot} + (A_1\Omega_1 + g_1)\alpha_{31} + (A_2\Omega_2 + g_2)\alpha_{32} + (A_3\Omega_3 + g_3)\alpha_{33} = \text{const} \qquad (1.7)$$

$$\alpha_{21}^2 + \alpha_{22}^2 + \alpha_{23}^2 = 1, \qquad \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^2 = 1 \qquad (1.8)$$

2. Let us establish some relations. By G_1, G_2, G_3 we denote the projections on the axes y_1, y_2, y_3 of the kinetic moment vector relative to the origin G of the coordinate system $Gy_1y_2y_3$ of the liquid particles in their motion about the center of mass G of the system; then,

$$G_1 = J_1\Omega_1 + g_1 \quad (123)$$

We make use of the transformations [1]

$$\omega_1^* = G_1 / J_1 \ (123), \qquad v_1 = u_1 + \Omega_2 y_3 - \Omega_3 y_2 + \omega_3^* y_2 - \omega_2^* y_3 \quad (123)$$

The expression for the kinetic energy T_2 of the liquid can be reduced to

$$2T_2 = M_2(r^{\cdot 2} + r^2\varphi^{\cdot 2} + \xi_3^{\cdot 2}) + \frac{G_1^2}{J_1} + \frac{G_2^2}{J_2} + \frac{G_3^2}{J_3} + \rho\int_\tau (v_1^2 + v_2^2 + v_3^2)\, d\tau$$

From (1.6) we infer that

$$T - U \leqslant T_0 - U_0 \qquad (T_0 = T|_{t=0},\ U_0 = U|_{t=0}) \qquad (2.1)$$

Finally, we can rewrite relation (2.1) and area integral (1.7) as

$$M(r^{\cdot 2} + r^2\varphi^{\cdot 2} + \xi_3^{\cdot 2}) + I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 + \frac{G_1^2}{J_1} + \frac{G_2^2}{J_2} + \frac{G_3^2}{J_3} +$$

$$+\rho\int_\tau (v_1^2+v_2^2+v_3^2)\,d\tau \leqslant \text{const}$$

$$Mr^2\varphi^{\cdot} + (I_1\Omega_1 + G_1)\alpha_{31} + (I_2\Omega_2 + G_2)\alpha_{32} + (I_3\Omega_3 + G_3)\alpha_{33} = \text{const}$$

respectively.

3. Eqs. (1.1) to (1.5) have the particular solution

$$\Omega_1 = \Omega_2 = 0, \quad \Omega_3 = \omega = \text{const}, \quad \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 0, \quad \alpha_{22} = \alpha_{33} = 1$$
$$g_1 = g_2 = g_3 = 0, \quad u_1 = u_2 = u_3 = 0, \quad r = r_0, \quad r^{\cdot} = 0, \quad \xi_3 = \xi_{30}$$
$$\xi_3^{\cdot} = 0, \quad \varphi^{\cdot} = \omega \tag{3.1}$$

where the quantities r_0 and ξ_{30} can be determined from Eqs.

$$\sum_{i=1}^{2}\sigma_i\left\{\frac{M}{R_{i0}^3} - \frac{15}{2R_{i0}^5}\left[\frac{A_2r_0^2}{R_{i0}^2} + \frac{A_3(\xi_3^{(i)}-\xi_{30})^2}{R_{i0}^2}\right] + \frac{3(A_1+A_2+A_3)}{2R_{i0}^5} - \frac{3A_3}{R_{i0}^5}\right\}(\xi^{(i)}-\xi_{30}) = 0$$
$$\left[\frac{\sigma_1(\xi_3^{(1)}-\xi_{30})}{R_{10}^5} + \frac{\sigma_2(\xi_3^{(2)}-\xi_{30})}{R_{20}^5}\right](A_3 - A_2) = 0 \tag{3.2}$$

$$R_{10}^2 = r_0^2 + (\xi_3^{(1)}-\xi_{30})^2, \qquad R_{20}^2 = r_0^2 + (\xi_3^{(2)}-\xi_{30})^2$$

and where the angular velocity ω of the center of mass of the system is given by Eq.

$$\omega^2 = -\frac{1}{Mr_0}\left(\frac{\partial U}{\partial r}\right)_0 = \sum_{i=1}^{2}\frac{\sigma_i}{MR_{i0}^3}\left\{M - \frac{15}{2R_{i0}^4}\left[A_2r_0^2 + A_3(\xi_3^{(i)}-\xi_{30})^2\right] + \right.$$
$$\left. + \frac{3(A_1+A_2+A_3)}{2R_{i0}^2} + \frac{3A_3}{2R_{i0}^2}\right\} \tag{3.3}$$

This particular solution corresponds to motion of the system in the circular orbit $r = r_0$, $\xi_3 = \xi_{30}$ with a constant angular velocity ω in such a way that the principal central axes y_1, y_2, y_3 of the ellipsoid of inertia of the system are directed along the tangent, principal normal, and binormal of the unperturbed orbit, respectively; the liquid is at rest relative to the solid, i.e. the system moves as a single solid body.

Let us investigate the stability of unperturbed motion (3.1) of the system with respect to the variables

$$\Omega_i,\ G_i,\ \alpha_{2i},\ \alpha_{3i},\ r,\ r^{\cdot},\ \xi_3,\ \xi_3^{\cdot},\ \varphi^{\cdot},\ \rho\int_\tau v_i^2\,d\tau \qquad (i = 1, 2, 3) \tag{3.4}$$

In perturbed motion we set

$$\Omega_3 = \omega + \Omega_3^*, \qquad G_3 = J_3\omega + G_3^*, \qquad \alpha_{22} = 1 + \alpha_{22}^*, \qquad \alpha_{33} = 1 + \alpha_{33}^*$$
$$r = r_0 + r^*, \qquad \xi_3 = \xi_{30} + \xi_3^*, \quad \varphi^{\cdot} = \omega + \omega^*$$

retaining the original notation for the remaining variables.

The equations of perturbed motion of the problem under consideration have the first integrals

$$W_1 = 4Mr_0\omega^2r^* + 2I_3\omega\Omega_3^* + 2\omega G_3^* + 2Mr_0^2\omega\omega^* + 6A_2P\alpha_{22}^* + 6A_3Q\alpha_{33}^* +$$
$$+ M(4r_0\omega r^*\omega^* + r_0^2\omega^{*2} + r^{\cdot 2} + \xi_3^{\cdot 2}) + 3P(A_1\alpha_{21}^2 + A_2\alpha_{22}^{*2} + A_3\alpha_{23}^2) +$$
$$+ 3Q(A_1\alpha_{31}^2 + A_2\alpha_{32}^2 + A_3\alpha_{33}^{*2}) + I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^{*2} +$$
$$+ \frac{G_1^2}{J_1} + \frac{G_2^2}{J_2} + \frac{G_3^{*2}}{J_3} + \rho\int_\tau (v_1^2 + v_2^2 + v_3^2)\,d\tau +$$
$$+ [M\omega^2 - (U_{rr})_0]r^{*2} - (U_{\xi_3\xi_3})_0\xi_3^{*2} - 2(U_{r\xi_3})_0r^*\xi_3^* - 2(U_{r\alpha_{22}})_0r^*\alpha_{22}^* - 2(U_{r\alpha_{33}})_0r^*\alpha_{33}^* -$$
$$- 2(U_{r\alpha_{23}})_0r^*\alpha_{23} - 2(U_{r\alpha_{32}})_0r^*\alpha_{32} - 2(U_{\xi_3\alpha_{22}})_0\xi_3^*\alpha_{22}^* - 2(U_{\xi_3\alpha_{33}})_0\xi_3^*\alpha_{33}^* -$$
$$- 2(U_{\xi_3\alpha_{23}})_0\xi_3^*\alpha_{23} - 2(U_{\xi_3\alpha_{32}})_0\xi_3^*\alpha_{32} + O(3) \leqslant \text{const}$$

$$W_2 = M(2r_0\omega r^* + r_0^2\omega^*) + I_3\Omega_3^* + G_3^* + A_3\omega\alpha_{33}^* +$$
$$+ M(\omega r^{*2} + 2r_0 r^*\omega^*) + (I_1\Omega_1 + G_1)\alpha_{31} + (I_2\Omega_2 + G_2)\alpha_{32} + (I_3\Omega_3^* + G_3^*)\alpha_{33}^* = \text{const}$$

$$W_3 = 2\alpha_{23}^* + \alpha_{21}^2 + \alpha_{23}^{*2} + \alpha_{23}^2 = 0, \qquad W_4 = 2\alpha_{33}^* + \alpha_{31}^2 + \alpha_{32}^2 + \alpha_{33}^{*2} = 0$$

Here

$$(U_{rr})_0 = \left(\frac{\partial^2 U}{\partial r^2}\right)_0 = -M\omega^2 + 3MP + \sum_{i=1}^{2}\sigma_i\left[-\frac{105}{2}A_2 r_0^4 R_{i0}^{-9} - \frac{105}{2}A_3 r_0^2(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-9} + \right.$$
$$\left. + 30A_2 r_0^2 R_{i0}^{-7} + \frac{15}{2}(A_1 + A_2 + A_3) r_0^2 R_{i0}^{-7}\right]$$

$$(U_{\xi_3\xi_3})_0 = \left(\frac{\partial^2 U}{\partial \xi_3^2}\right)_0 = -M\omega^2 + 3MQ + \sum_{i=1}^{2}\sigma_i\left[-\frac{105}{2}A_2 r_0^2(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-9} - \right.$$
$$-\frac{105}{2}A_3 r_0^2(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-9} + 3(A_2 - A_3)R_{i0}^{-5} + \frac{15}{2}(A_1 + A_2 + A_3)(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-7} +$$
$$\left. + 30A_3(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-7}\right]$$

$$(U_{r\xi_3})_0 = \left(\frac{\partial^2 U}{\partial r\,\partial \xi_3}\right)_0 = \sum_{i=1}^{2}\sigma_i\left[\frac{105}{2}A_2 r_0^3(\xi_3^{(i)} - \xi_{30})R_{i0}^{-9} + \frac{105}{2}A_3 r_0(\xi_3^{(i)} - \xi_{30})^3 R_{i0}^{-9} - \right.$$
$$\left. -\frac{15}{2}(A_1 + A_2 + A_3) r_0(\xi_3^{(i)} - \xi_{30})R_{i0}^{-7} - 15(A_2 + A_3) r_0(\xi_3^{(i)} - \xi_{30})R_{i0}^{-7}\right] \qquad (3.5)$$

$$(U_{r\alpha_{22}})_0 = \left(\frac{\partial^2 U}{\partial r\,\partial \alpha_{22}}\right)_0 = 3A_2 r_0 \sum_{i=1}^{2}\sigma_i(5r_0^2 R_{i0}^{-2} - 2)R_{i0}^{-5}$$

$$(U_{r\alpha_{23}})_0 = \left(\frac{\partial^2 U}{\partial r\,\partial \alpha_{23}}\right)_0 = 15A_3 r_0^2 \sum_{i=1}^{2}\sigma_i(\xi_{30} - \xi_3^{(i)})R_{i0}^{-7}$$

$$(U_{r\alpha_{32}})_0 = \left(\frac{\partial^2 U}{\partial r\,\partial \alpha_{32}}\right)_0 = 15A_2 r_0^2 \sum_{i=1}^{2}\sigma_i(\xi_3^{(i)} - \xi_{30})R_{i0}^{-7}$$

$$(U_{r\alpha_{33}})_0 = \left(\frac{\partial^2 U}{\partial r\,\partial \alpha_{33}}\right)_0 = 15A_3 r_0 \sum_{i=1}^{2}\sigma_i(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-7}$$

$$(U_{\xi_3\alpha_{22}})_0 = \left(\frac{\partial^2 U}{\partial \xi_3\,\partial \alpha_{22}}\right)_0 = 15A_2 r_0^2 \sum_{i=1}^{2}\sigma_i(\xi_{30} - \xi_3^{(i)})R_{i0}^{-7}$$

$$(U_{\xi_3\alpha_{23}})_0 = \left(\frac{\partial^2 U}{\partial \xi_3\,\partial \alpha_{23}}\right)_0 = 3A_3 r_0 \sum_{i=1}^{2}\sigma_i[1 + 5(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-2}]R_{i0}^{-5}$$

$$(U_{\xi_3\alpha_{32}})_0 = \left(\frac{\partial^2 U}{\partial \xi_3\,\partial \alpha_{32}}\right)_0 = 3A_2 r_0 \sum_{i=1}^{2}\sigma_i[1 - 5(\xi_3^{(i)} - \xi_{30})^2 R_{i0}^{-2}]R_{i0}^{-5}$$

$$(U_{\xi_3\alpha_{33}})_0 = \left(\frac{\partial^2 U}{\partial \xi_3\,\partial \alpha_{33}}\right)_0 = 15A_3 \sum_{i=1}^{2}\sigma_i(\xi_{30} - \xi_3^{(i)})^3 R_{i0}^{-7}$$

$$P = r_0^2(\sigma_1 R_{10}^{-5} + \sigma_2 R_{20}^{-5}), \qquad Q = \sigma_1(\xi_3^{(1)} - \xi_{30})^2 R_{10}^{-5} + \sigma_2(\xi_3^{(2)} - \xi_{30})^2 R_{20}^{-5}$$

and $O(3)$ denotes terms of not lower than the third order of smallness with respect to the perturbations. With perturbed motion, by virtue of (1.6), $dW_1/dt \leq 0$.

Let us consider the following function of the variables of the problem constructed by Chetaev's method [4] as a bundle of the first integrals of the equations of motion,

$$W = W_1 - 2\omega W_2 - 3A_2 P W_3 + A_2(\omega^2 - 3Q) W_4 + \lambda W_2^3 + {}^1/_4\, \nu W_3^2 + {}^1/_4\, \varkappa W_4^2 =$$
$$= W^{(1)} + W^{(2)} + W^{(3)} + O(3) \tag{3.6}$$

Here

$$W^{(1)} = M(r'^2 + \xi_3'^2) + 3(A_1 - A_2) P\alpha_{21}^2 + \rho \int_\tau (v_1^2 + v_2^2 + v_3^2)\, d\tau$$

$$W^{(2)} = I_1\Omega_1^2 + G_1^2 / J_1 + [A_3\omega^2 - 3(A_3 - A_1) Q]\alpha_{31}^2 - 2I_1\omega\Omega_1\alpha_{31} - 2\omega G_1\alpha_{31}$$

$$\begin{aligned} W^{(3)} = {} & I_2\Omega_2^2 - 2I_2\omega\Omega_2\alpha_{32} + I_3(1 + \lambda I_3)\Omega_3^{*2} + 2\lambda I_3\Omega_3^* G_3^* + 2I_3\omega(\lambda A_3 - 1)\Omega_3^*\alpha_{33}^* + \\ & + 2\lambda M I_3 r_0^2 \Omega_3^*\omega^* + 4\lambda M I_3 r_0 \omega\Omega_3^* r^* + G_2^2 / J_2 - 2\omega G_2\alpha_{32} + (1 + \lambda J_3) G_3^{*2} / J_3 + \\ & + 2\omega(\lambda A_3 - 1) G_3^*\alpha_{33}^* + 2\lambda M r_0^2 G_3^*\omega^* + 4\lambda M r_0\omega G_3^* r^* + \nu\alpha_{33}^{*2} - 2(U_{\xi_3\alpha_{33}})_0 \xi_3^*\alpha_{33}^* - \\ & - 2(U_{r\alpha_{33}})_0 r^*\alpha_{33}^* + 3(A_3 - A_3) P\alpha_{23}^2 - 2(U_{\xi_3\alpha_{23}})_0 \xi_3^*\alpha_{23} - 2(U_{r\alpha_{23}})_0 r^*\alpha_{23} + \\ & + [A_3\omega^2 - 3(A_3 - A_2) Q]\alpha_{32}^2 - 2(U_{\xi_3\alpha_{32}})_0 \xi_3^*\alpha_{32} - 2(U_{r\alpha_{32}})_0 r^*\alpha_{32} + (A_3\omega^2 + \lambda A_3^2\omega^2 + \\ & + \varkappa)\alpha_{33}^{*2} + 2\lambda M A_3 r_0^2 \omega\omega^*\alpha_{33}^* - 2(U_{\xi_3\alpha_{33}})_0 \xi_3^*\alpha_{33}^* + 2[2\lambda M A_3 r_0\omega^2 - (U_{r\alpha_{33}})_0] r^*\alpha_{33}^* + \\ & + M r_0^2 (1 + \lambda M r_0^2)\omega^{*2} + 4\lambda M^2 r_0^3 \omega\omega^* r^* - (U_{\xi_3\xi_3})_0 \xi_3^{*2} - 2(U_{r\xi_3})_0 \xi_3^* r^* + \\ & + [4\lambda M^2 r_0^2\omega^2 - M\omega^2 - (U_{rr})_0] r^{*2} \end{aligned}$$

and λ, ν, $\varkappa$ are sufficiently large positive quantities chosen from the condition of positive definition of the function W.

By the Sylvester criterion the quadratic forms $W^{(1)}$, $W^{(2)}$, and $W^{(3)}$, and hence the function W, are positively defined if and only if

$$(A_1 - A_2)\, P > 0, \qquad (A_3 - A_1)(\omega^2 - 3Q) > 0$$
$$(A_3 - A_2) P > 0, \qquad (A_3 - A_2)\cdot(\omega^2 - 3Q) > 0 \tag{3.7}$$

$$3(A_3 - A_2)(\omega^2 - 3Q) P (U_{\xi_3\xi_3})_0 + 3P(U_{\xi_3\alpha_{31}})_0 (U_{r\alpha_{31}})_0 + (\omega^2 - 3Q)(U_{\xi_3\alpha_{23}})_0^2 < 0 \tag{3.8}$$

$$(M r_0^2 + A_3)(\Phi_{11}^\circ \Phi_{22}^\circ - \Phi_{12}^{\circ 2}) + 4M^2 r_0^2 \omega^2 \Phi_{11}^\circ > 0 \tag{3.9}$$

Here

$$\Phi_{11}^\circ = -(U_{\xi_3\xi_3})_0 - [3(A_3 - A_2) P]^{-1} (U_{\xi_3\alpha_{23}})_0 - [(A_3 - A_3)(\omega^2 - 3Q)]^{-1} (U_{\xi_3\alpha_{31}})_0^2$$

$$\begin{aligned}\Phi_{12}^\circ = {} & -(U_{r\xi_3})_0 - [3(A_3 - A_2) P]^{-1} (U_{r\alpha_{23}})_0 (U_{\xi_3\alpha_{23}})_0 - \\ & -[(A_3 - A_2)(\omega^2 - 3Q)]^{-1} (U_{r\alpha_{31}})_0 (U_{\xi_3\alpha_{31}})_0\end{aligned}$$

$$\Phi_{22}^\circ = -[M\omega^2 + (U_{rr})_0] - [3(A_3 - A_2) P]^{-1} (U_{r\alpha_{23}})_0^2 - [(A_3 - A_1)(\omega^2 - 3Q)]^{-1} (U_{r\alpha_{31}})_0^2$$

If conditions (3.7) to (3.9) are fulfilled the function W given by Formula (3.6) is the Liapunov function for our problem. In fact, the dW/dt chosen on the basis of the equations of perturbed motion is nonpositive by virtue of (1.6). From this, by Rumiantsev's theorem on stability with respect to some of the variables [5], we infer that unperturbed motion (3.1) of a solid with a cavity filled with a viscous incompressible liquid is stable with respect to quantities (3.4). As we know, the unperturbed motion of system (3.1) can be regarded as the resultant of two motions: the motion of the center of mass and the motion about the center of mass. Conditions (3.7) in this case are the conditions of stability of motion of our system about the center of mass, while (3.8) and (3.9) are the conditions of stability of motion of its center of mass.

We note that if the total mass of the system is considered as concentrated in its center of mass, then conditions (3.8) and (3.9) become the familiar criterion [4] of the stability of circular orbits of a material point in an axisymmetric force field with the force function $U = U(r, \xi_3)$,

$$(U_{\xi_3\xi_3})_0 < 0, \qquad \left[(U_{rr})_0 + \frac{3}{r_0}(U_r)_0\right](U_{\xi_3\xi_3})_0 - (U_{r\xi_3})_0^2 > 0$$

In conclusion we note that in the case of a single attracting center conditions (3.7) become the familiar [2 and 3] conditions $A_3 > A_1 > A_2$.

4. Example. Let us consider the stability of motion of a spacecraft with a cavity completely filled with a viscous liquid in the Earth's normal gravitational field.

As we know [6], the potential energy of a point of unit mass attracted by two stationary centers of equal masses $M/2$ lying at the distance $2ci$ ($i = \sqrt{-1}$) from each other is given by

$$\Pi = -\frac{1}{2} fM \left\{ \frac{1}{\sqrt{\xi_1^2 + \xi_2^2 + (\xi_3 - ic)^2}} + \frac{1}{\sqrt{\xi_1^2 + \xi_2^2 + (\xi_3 + ic)^2}} \right\} \qquad (4.1)$$

and can be expressed as a series in Legendre polynomials,

$$\Pi = -\frac{fM}{r} \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{c}{r}\right)^{2n} P_{2n}\left(\frac{\xi_3}{r}\right) \right\} \qquad (r^2 = \xi_1^2 + \xi_3^2)$$

Setting M equal to the Earth's mass and $c \approx 210$ km, we find that the first two terms of series (4.1) are equal to the corresponding terms of the expansion of the Earth's potential in a series in Legendre polynomials, and that the third terms of these series are also sufficiently close. Hence, expansion (4.1) is an adequate representation of the potential of the Earth's normal field.

If we set

$$\xi_3^{(1)} = -ic, \quad \xi_3^{(2)} = ic, \; m_1 = m_2 = M/2, \; f_1 = f_2 = f$$

we find that Eqs. (3.2) have the solution $\xi_3 = \xi_{30} = 0$, $r = r_0$, where the angular velocity ω of the center of mass of the spacecraft is related to r_0 by Expression (3.3), which in this case becomes

$$\omega^2 = \frac{f}{R_0^3}\left[M - \frac{15}{2R_0^4}(A_2 r_0^2 - A_3 c^2) + \frac{3(A_1 + A_2 + A_3)}{2R_0^2} + \frac{3A_2}{R_0^2} \right] \qquad (R_0^2 = r_0^2 - c^2)$$

Solution (3.1) corresponds to motion of the spacecraft in the circular orbit $r = r_0$, $\xi_3 = 0$ in the plane of the equator with the constant angular velocity ω in such a way that the principal central axes y_1, y_2, y_3 of the ellipsoid of inertia of the dynamic system are directed along the tangent, principal normal, and binormal of the unperturbed orbit, respectively; the liquid is at rest relative to the wall of the spacecraft cavity, i.e. the entire system moves as a single solid body.

For the quantities P and Q introduced above by means of Formulas (3.5) we have Expressions

$$P = r_0^2 (f_1 m_1 R_{10}^{-5} + f_2 m_2 R_{20}^{-5}) = \frac{fMr_0^2}{(r_0^2 - c^2)^{5/2}} > 0$$

$$Q = f_1 m_1 (\xi_3^{(1)} - \xi_{30})^2 R_{10}^{-5} + f_2 m_2 (\xi_3^{(2)} - \xi_{30})^2 R_{20}^{-5} = -\frac{fMc^2}{(r_0^2 - c^2)^{5/2}} < 0$$

Conditions (3.7) then give us

$$A_3 > A_1 > A_2 \qquad (4.2)$$

Further, since the ratio $c^2/(r_0^2 - c^2)$ is very small as compared with unity, condition (3.8) can be written approximately either as $(U_{\xi_3\xi_3})_0 < 0$ or as $-(\omega^2 - 3Q)M < 0$. Since $Q < 0$, this condition is, in fact, fulfilled. For the same reasons condition (3.9) can be rewritten approximately as

$$[(U_{rr})_0 + 3/r_0 (U_r)_0](U_{\xi_3\xi_3})_0 - (U_{r\xi_3})_0^2 > 0$$

By virtue of (3.5) the second term in this expression is considerably smaller than the first, so that (approximately) $[(U_{rr})_0 + 3/r_0 (U_r)_0](U_{\xi_3\xi_3})_0 > 0$. Since $(U_{\xi_3\xi_3})_0 < 0$, it follows that

$$(U_{rr})_0 + \frac{3}{r_0}(U_r)_0 = -\frac{fM^2}{(r_0^2 - c^2)^{3/2}}\left(4 - \frac{3r_0^2}{r_0^2 - c^2}\right) < 0$$

Since $c \approx 210$ km, and since r_0 is larger than the Earth's radius, this condition is fulfilled for all real satellite motions of spacecraft.

Thus, we have shown that stability conditions (3.7) to (3.9) for unperturbed motion (3.1) of a spacecraft can be reduced to conditions (4.2). This proves that the stability conditions for the described motion of a spacecraft in the Earth's normal gravitational field and for its motion in the Earth's central force field are of the same form [2 and 3].

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